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Chaotic Actions of Locally Compact Hausdorff Topological Groups

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Abstract

In this paper we study continuous actions of topological groups. We introduce a parametrized notion of periodicity – relative to a fixed class of compactifications of the acting group. This yields a natural generalization of Devaney's well-recognized concept of chaos. As our main result, we establish a geometric characterization of those classes of compactifications of a locally compact Hausdorff topological group for which the group admits a faithful chaotic continuous action on some (compact) Hausdorff space.

Keywords: topological dynamics, topological group, continuous group action, periodicity, chaos.

1 Introduction

Since the 1970s, chaotic dynamics has attracted a great deal of attention within the theory of dynamical systems (see, e.g., [12]), and much effort has been put into giving the notion of chaos a precise mathematical definition. In this regard, several concepts of chaos have been established, and there is an extensive literature dealing with the interconnections between all the different approaches. For a detailed exposition of this issue we refer to [9,14].

Among the best-recognized definitions of chaos is the one given by R. Devaney in [7], which applies to classical discrete- and real-time dynamical systems. In [17], Devaney's concept is extended to the general setting of continuous semigroup actions, and two notions, referred to as *chaos* and *strong chaos*, are developed in terms of syndeticity in topological semigroups (cf. [10]). These coincide for unitary actions of commutative topological monoids and discrete groups (the former is explained in [17], the latter in [18]). Moreover, generalizing a classical result by Banks et al. ([2]),

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it is shown in [17] that any chaotic continuous action on a Hausdorff uniform space has sensitive dependence on initial conditions.

In [18] the notion of strong chaos is explored in the slightly more specific but still wide-ranging context of topological groups. Among other things it is shown there that a locally compact Hausdorff topological group admits a faithful strongly chaotic continuous action on some Hausdorff space if and only if it is strongly syndetically separated and non-compact. In the case where the acting group is discrete this result specializes to a characterization of infinite residually finite groups in terms of topological dynamics, which is due to [4]. As one might expect from this, the concept of chaos developed in [4], which has been discussed by several authors, particularly from a group-theoretic point of view, e.g. in [5,15,16], is closely related to the one established in [17] (see Section 4 of this paper).

The aim of the present paper is to introduce a parametrized version of the term *periodicity* relative to a class of compactifications of the acting group and to characterize those classes of compactifications of a fixed topological group for which there exists a faithful respectively chaotic continuous action on some (compact) Hausdorff space.

The paper is organized as follows. In Section 2 we address certain rather general issues concerning continuous group actions. Afterwards, in Section 3, we introduce some terminology regarding compactifications of topological groups and a closely related kind of separation property. In Section 4 we establish the desired notions of periodicity and chaos relative to a fixed class of group compactifications and illustrate how these generalize the corresponding concepts discussed in [17,18]. In Section 5, we provide a geometric characterization of those classes of compactifications of a fixed locally compact Hausdorff topological group G for which G admits a faithful respectively chaotic continuous action on some (compact) Hausdorff space (see Theorem 5.5). As a special instance of this parametrized result, we obtain a novel description of non-compact, maximally almost periodic, locally compact Hausdorff topological groups by means of topological dynamics (see Corollary 5.6).

2 Preliminaries

In this section we establish some terminology and recall several well-known concepts concerning topological groups and their continuous actions, which shall be essential towards the aim of this article. For a start, let us briefly fix some notation regarding continuous real-valued functions.

Let X be a topological space. If Y is another topological space, then we denote by $C(X,Y)$ the set of all continuous maps from X into Y. Let $C(X) := C(X,\mathbb{R})$. For $r \in \mathbb{R}$, let $r_X \colon X \to \mathbb{R}, x \mapsto r$. For $f \in C(X)$, we define $|f| \colon X \to \mathbb{R}, x \mapsto |f(x)|$. Furthermore, we denote by $C_b(G)$ the set of all bounded functions among $C(X)$. As usual, we obtain a normed $\mathbb{R}$-vector space by equipping $C_b(X)$ with the obvious point-wise operations and the norm given by

$$\|f\|_\infty := \sup_{x\in X} |f(x)| \quad (f \in C_b(X)).$$

If f is any element of $C(X)$, then we call $\operatorname{spt} f := \overline{\{x \in G \mid f(x) \neq 0\}}$ the *support of* f. We shall consider the set $C_c(X) := \{f \in C(X) \mid \operatorname{spt} f \text{ is compact in } X\}$, which evidently constitutes a linear subspace of $C_b(X)$.

Let us recall some notation concerning normed vector spaces. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let X be a normed $\mathbb{K}$-vector space. We denote by $B(X)$ the closed unit ball of X. Furthermore, the *dual space of* X, i.e., the normed $\mathbb{K}$-vector space of all continuous linear functionals on X equipped with the operator norm, shall be denoted by X'.

Next we address some notational issues about topological groups. Let G be a topological group, i.e., G is supposed to be a group equipped with a topology such that the map $G \times G \to G$, $(x, y) \mapsto x^{-1}y$ is continuous. We denote by e_G the neutral element of G. Furthermore, we shall occasionally refer to the continuous maps given by $\rho_G(g)\colon G \to G$, $x \mapsto xg$ where $g \in G$. If N is a subgroup of G, then we turn $G/N := \{gN \mid g \in G\}$ into a topological space by endowing it with the quotient topology generated by the map $\pi_N\colon G \to G/N$, $g \mapsto gN$.

Finally in this section, we want to introduce some very basic concepts of topological dynamics as they may be found in standard textbooks such as [8]. Let X be a topological space, let G be a topological group, and let α be a *continuous action of* G *on* X, i.e., α is a continuous map from $G \times X$ to X such that $\alpha(e_G, x) = x$ and $\alpha(gh, x) = \alpha(g, \alpha(h, x))$ for all $x \in X$, $g, h \in G$. As usual, α shall be called *faithful* if, for every $g \in G \setminus \{e_G\}$, there exists $x \in X$ such that $\alpha(g, x) \neq x$. We say that α is *topologically transitive* if, for any two non-empty open subsets $U, V \subseteq X$, there exists $g \in G$ such that $\alpha(g, U) \cap V \neq \emptyset$. Furthermore, α is called *minimal* if $\alpha(G, x)$ is dense in X for every $x \in X$.

The following two lemmata are quite elementary and folklore. Whereas the former will turn out to be useful in the proof of Proposition 4.6, the latter is connected to Theorem 5.5 rather directly.

Lemma 2.1 (e.g. [18]) *Let G be a topological group and α a continuous action of G on some topological space X. Then α is topologically transitive if and only if, for every finite non-void family of open non-empty subsets $U_0, \ldots, U_{n-1} \subseteq X$, there exist an open non-empty subset $V \subseteq X$ as well as $g_0, \ldots, g_{n-1} \in G$ such that $\alpha(g_i, V) \subseteq U_i$ for all $i \in \{0, \ldots, n-1\}$.*

Lemma 2.2 (e.g. [8]) *Let G be a topological group and α a continuous action of G on some Hausdorff space X. If G is compact and α is topologically transitive, then α is minimal.*

Let G be a topological group and let α be a continuous action of G on some compact Hausdorff space X. Then the action α is called *equicontinuous* if the set $\{x \mapsto \alpha(g, x) \mid g \in G\}$ is equicontinuous with respect to the unique uniform structure generating the topology of X, that is, α is equicontinuous if and only if, for every $x \in X$ and every open neighborhood V of the diagonal $\Delta_X := \{(y, y) \mid y \in X\}$ in $X \times X$, there exists some open neighborhood U of x in X such that

$$\forall g \in G \, \forall y \in U : (\alpha(g, x), \alpha(g, y)) \in V.$$

Moreover, α is called *Kronecker* if α is minimal and equicontinuous. As would seem natural, a point $x \in X$ is called *Kronecker with respect to* α if the restricted action

$$\alpha_x \colon \ G \times \overline{\alpha(G, x)} \to \overline{\alpha(G, x)}, \ (g, y) \mapsto \alpha(g, y)$$

is Kronecker.

3 Group compactifications and separation properties

In this section we want to discuss separation properties in topological groups arising from certain compactifications and draw a connection to aspects of topological dynamics. For a more general approach to compactifications of topological groups by right topological semigroups, which shall not be needed in the course of this article, we refer to [3].

Definition 3.1 Let G be a topological group. A *group compactification of* G is a pair (H, h) consisting of a compact Hausdorff topological group H and a continuous homomorphism $h \colon G \to H$ such that $h[G]$ is dense in H. A *morphism* from a group compactification (H_0, h_0) of G into another one (H_1, h_1) is a continuous homomorphism $f \colon H_0 \to H_1$ where $f \circ h_0 = h_1$. The corresponding category of group compactifications of G shall be denoted by $\mathcal{C}(G)$.

Remark 3.2 Let G be a topological group. Since any group compactification of G factors by a quotient map through the Stone-Čech compactification of the underlying topological space of G, the category $\mathcal{C}(G)$ is essentially small, i.e., the class of isomorphism classes of objects in $\mathcal{C}(G)$ is a set.

Our next objective is to introduce a family of certain geometric separation properties for topological groups and to mention some simple reformulations.

Definition 3.3 Let G be a topological group and $\mathcal{H}$ a set of subgroups of G. We say that $\mathcal{H}$ *separates compact subsets in* G if, for every compact subset $K \subseteq G \setminus \{e_G\}$, there exists some $H \in \mathcal{H}$ such that $H \cap K = \emptyset$. A subclass $\mathcal{D} \subseteq \mathcal{C}(G)$ is said to *separate compact subsets in* G if $\{\ker(h) \mid (H, h) \in \mathcal{D}\}$ separates compact subsets in G.

Proposition 3.4 *Let G be a topological group and $\mathcal{H}$ a set of subgroups of G. The following are equivalent:*

(a) $\mathcal{H}$ separates compact subsets in G.

(b) For any two disjoint compact subsets $A, B \subseteq G$, there exists some $H \in \mathcal{H}$ such that $\pi_H[A] \cap \pi_H[B] = \emptyset$.

(c) For any finite family of pairs of compact subsets $K_0, K_0', \ldots, K_{n-1}, K_{n-1}' \subseteq G$ where $K_i \cap K_i' = \emptyset$ for each $i \in \{0, \ldots, n-1\}$, there exists $H \in \mathcal{H}$ such that $\pi_H[K_i] \cap \pi_H[K_i'] = \emptyset$ for all $i \in \{0, \ldots, n-1\}$.

Proof. Of course, (c)$\Longrightarrow$(b)$\Longrightarrow$(a). In order to show that (a)$\Longrightarrow$(c), consider compact subsets $K_0, K_0', \ldots, K_{n-1}, K_{n-1}' \subseteq G$ such that $K_i \cap K_i' = \emptyset$ for each

$i \in \{0, \ldots, n-1\}$. We observe that $K := \bigcup_{i=0}^{n-1} K_i^{-1} K_i'$ is a compact subset of G, and K does not contain e_G. By (a), there exists some $H \in \mathcal{H}$ such that $H \cap K = \emptyset$, that is, $e_H \notin \pi_H[K]$. Hence, it follows that $\pi_H[K_i] \cap \pi_H[K_i'] = \emptyset$ for each $i \in \{0, \ldots, n-1\}$. $\square$

The subsequent result relates the separation property introduced in Definition 3.3 to aspects of topological dynamics. In fact, Proposition 3.5 will turn out to be quite useful when proving that a particular class of continuous group actions satisfies a certain density condition (see Proposition 5.4).

Proposition 3.5 *Let G be a Hausdorff topological group. A subclass $\mathcal{D} \subseteq \mathcal{C}(G)$ separates compact subsets in G if and only if $\{p \circ h \mid p \in C(H, [-1,1]),\ (H,h) \in \mathcal{D}\}$ is dense in $C(G, [-1,1])$ with respect to the topology of compact convergence.*

Proof. "$\Longrightarrow$" Let $f \in C(G, [-1,1])$, consider a non-empty compact subset $K \subseteq G$, and let $\varepsilon > 0$. Since f is continuous, $\mathcal{U} := \{U \mid \emptyset \neq U \subseteq G \text{ open}, \operatorname{diam} f[U] \leq \varepsilon\}$ constitutes an open cover of G. Due to compactness of K, there is a non-empty finite subset $\mathcal{V} \subseteq \mathcal{U}$ such that $K \subseteq \bigcup \mathcal{V}$. For every $U \in \mathcal{V}$,

$$V(U) := f^{-1}[(\inf f[U] - \varepsilon/2, \sup f[U] + \varepsilon/2)]$$

is an open neighborhood of $\overline{U}$ in G because $f[\overline{U}] \subseteq \overline{f[U]} \subseteq [\inf f[U], \sup f[U]]$. By Proposition 3.4, there is some $(H,h) \in \mathcal{D}$ such that $h[K \cap \overline{U}] \cap h[K \setminus V(U)] = \emptyset$ for all $U \in \mathcal{V}$. Using the Urysohn Lemma, we conclude that, for each $U \in \mathcal{V}$, there exists a continuous function $p_U \colon H \to [-1, (\inf f[U] + \sup f[U])/2]$ such that $p_U(x) = (\inf f[U] + \sup f[U])/2$ for all $x \in h[K \cap \overline{U}]$ and $p_U(x) = -1$ for all $x \in h[K \setminus V(U)]$. Let us define $p \in C(H, [-1,1])$ by $p(x) := \sup_{U \in \mathcal{V}} p_U(x)$ for all $x \in H$. We show that $\sup_{x \in K} |f(x) - p(h(x))| \leq \varepsilon$. For this purpose, let $x \in K$. Then there exists some $U \in \mathcal{V}$ such that $x \in U$, which implies that

$$f(x) \leq (\inf f[U] + \sup f[U])/2 + \varepsilon/2 = p_U(h(x)) + \varepsilon/2 \leq p(h(x)) + \varepsilon.$$

Moreover, for each member $U \in \mathcal{V}$, it follows that either $x \in V(U)$ and hence $p_U(h(x)) \leq (\inf f[U] + \sup f[U])/2 \leq f(x) + \varepsilon$, or $x \in K \setminus V(U)$ and therefore $p_U(h(x)) = -1 \leq f(x) + \varepsilon$. This shows that $p(h(x)) \leq f(x) + \varepsilon$ as well. Consequently, $|f(x) - p(h(x))| < \varepsilon$. So, $\sup_{x \in K} |f(x) - p(h(x))| \leq \varepsilon$.

"$\Longleftarrow$" Let $K \subseteq G \setminus \{e_G\}$ be compact. Since G is a completely regular Hausdorff space (see [1]), there exists a function $f \in C(G, [-1,1])$ such that $f(e_G) = 1$ and $f[K] \subseteq \{-1\}$. By assumption, there exist $(H,h) \in \mathcal{D}$ and $p \in C(H, [-1,1])$ where $\sup_{x \in K \cup \{e_G\}} |f(x) - p(h(x))| < 1$. Therefore $(p \circ h)[\ker(h)] = \{(p \circ h)(e_G)\} \in [-1, 0)$ and $(p \circ h)[K] \subseteq (0, 1]$, and consequently $\ker(h) \cap K = \emptyset$. $\square$

We finish this section by having a closer look at the particular instances of Definition 3.3 with regard to certain classes of group compactifications. A short moment of reflection readily yields the following simple observation:

Proposition 3.6 *Let G be a topological group. The following are equivalent:*

(a) *G is maximally almost periodic, i.e., there exists a one-to-one continuous homomorphism from G into a compact Hausdorff topological group.*

(b) *$\{N \trianglelefteq G \mid G/N$ is maximally almost periodic$\}$ separates compact subsets in G.*

(c) *$\mathcal{C}(G)$ separates compact subsets in G.*

Proof. Evidently, (a) implies (b), and the latter is equivalent to (c). So, it remains to substantiate that (c) implies (a). To this end, choose a subset $\mathcal{D} \subseteq \mathcal{C}(G)$ separating compact subsets in G, and consider the compact Hausdorff topological group $K := \prod_{(H,h)\in\mathcal{D}} H$. We need to argue that the continuous homomorphism $k\colon G \to K$, $g \mapsto (h(g))_{(H,h)\in\mathcal{D}}$ is indeed one-to-one. However, this is obvious: for every $g \in G \setminus \{e_G\}$, there exists some $(H,h) \in \mathcal{D}$ such that $g \notin \ker(h)$, which means that $h(g) \neq e_H$ and hence $k(g) \neq e_K$. □

Considering another class of compactifications of a given topological group, we obtain a concept studied in [18].

Definition 3.7 Let G be a topological group. A subset $H \subseteq G$ is called *syndetic in G* if there exists a compact subset $K \subseteq G$ such that $KH = G$. Furthermore, G is called *strongly syndetically separated* if the set of all syndetic closed normal subgroups of G separates compact subsets in G, or equivalently, if $\mathcal{Q}(G) := \{(H,h) \in \mathcal{C}(G) \mid \ker(h)$ is syndetic in $G\}$ separates compact subsets in G.

In order to give a better understanding of the class of group compactifications introduced above, we present the following fact, which is both basic and folklore:

Proposition 3.8 (e.g. [18]) *Let G be a topological group and $H \leq G$. If H is syndetic in G, then G/H is compact. Conversely, if G is locally compact Hausdorff and G/H is compact, then H is syndetic in G.*

Corollary 3.9 *Let G be a topological group and let $(H,h) \in \mathcal{C}(G)$. If $\ker(h)$ is syndetic in G, then $h\colon G \to H$ is a quotient map. Conversely, if G is locally compact Hausdorff and $h\colon G \to H$ is a quotient map, then $\ker(h)$ is syndetic in G.*

Let us provide a simple example of a strongly syndetically separated topological group.

Example 3.10 [[18]] $\mathbb{R}^n$ is strongly syndetically separated.

Proof. Let $K \subseteq \mathbb{R}^n \setminus \{0\}$ be a compact subset. For each $i \in \{0,\ldots,n-1\}$, there exists $t_i > 0$ such that $\mathrm{pr}_i[K] \subseteq (-t_i,t_i)$. Consider the subgroup $H := \prod_{i=0}^{n-1} \mathbb{Z}\cdot t_i$ of $\mathbb{R}^n$. Note that H is closed in $\mathbb{R}^n$ as $\mathbb{Z}\cdot t_i$ is closed in $\mathbb{R}$ for each $i \in \{0,\ldots,n-1\}$. Furthermore, H is syndetic in $\mathbb{R}^n$ because $C := \prod_{i=0}^{n-1}[0,t_i]$ is compact in $\mathbb{R}^n$ and $C + H = \mathbb{R}^n$. Moreover, $H \cap K = \emptyset$ by choice of $t_0,\ldots,t_{n-1}$ and our hypothesis that $0 \notin K$. Thus, $\mathbb{R}^n$ is strongly syndetically separated. □

For the case of discrete topological groups, we easily conclude the following:

Remark 3.11 [[18]] Let G be a discrete topological group. A subgroup H of G is syndetic in G if and only if G/H is finite. Consequently, G is strongly syndetically separated if and only if G is residually finite.

4 Periodicity and chaos

In this section we want to discuss the notions of periodicity and chaos in a comparably broad sense. In [17,18] a vast generalization of Devaney's notion of chaos to the setting of continuous semigroup actions has been established and investigated. However, in this article we want to introduce these concepts in a parametrized manner: in fact, we want introduce a version of periodicity relative to a class of group compactifications as considered in Section 3. This will significantly improve the applicability of our results.

Definition 4.1 Let G be a topological group and α a continuous action of G on some topological space X. Moreover, let us consider a subclass $\mathcal{D} \subseteq \mathcal{C}(G)$. Then α is said to be $\mathcal{D}$-*periodic* if there exist $(H, h) \in \mathcal{D}$ and a continuous action β of H on X such that $\alpha(g, x) = \beta(h(g), x)$ for all $g \in G$ and $x \in X$. A point $x \in X$ is said to be $\mathcal{D}$-*periodic with respect to* α if the restricted action

$$\alpha_x\colon\ G \times \overline{\alpha(G, x)} \to \overline{\alpha(G, x)},\ (g, y) \mapsto \alpha(g, y)$$

is $\mathcal{D}$-periodic. Finally, α is called $\mathcal{D}$-*chaotic* if

(1) α is topologically transitive,

(2) the set of points being $\mathcal{D}$-periodic with respect to α is dense in X,

(3) α is not minimal.

As in Section 3, we want to investigate some particular instances of the introduced notions. To start with, let us briefly discuss the connection to the Kronecker property. The following note is an immediate consequence the Arzela-Ascoli Theorem.

Proposition 4.2 *Let G be a topological group and α a continuous action of G on some compact Hausdorff space X. The action α is $\mathcal{C}(G)$-periodic if and only if α is Kronecker.*

Proof. "$\Longrightarrow$" By assumption, there exists some $(H, h) \in \mathcal{C}(G)$ as well as a continuous action β of H on X such that $\alpha(g, x) = \beta(h(g), x)$ for all $g \in G$ and $x \in X$. Since β is equicontinuous by the Arzela-Ascoli Theorem and furthermore $\{x \mapsto \alpha(g, x) \mid g \in G\} = \{x \mapsto \beta(h(g), x) \mid g \in G\} \subseteq \{x \mapsto \beta(g, x) \mid g \in H\}$, we conclude that α is equicontinuous. Besides, due to density of $h[G]$ in H, it follows that $\overline{\alpha(G, x)} = \overline{\beta(H, x)} = X$ for all $x \in X$. Hence, α is minimal and therefore Kronecker.

"$\Longleftarrow$" Let H denote the closure of $J := \{x \mapsto \alpha(g, x) \mid g \in G\}$ in $C(X, X)$ with respect to the compact-open topology, and let us endow H with the respective subspace topology. Due to the Arzela-Ascoli Theorem, H is compact. Since H is a topological monoid and J is a subset of invertible elements being dense in H, we conclude that H is a group and hence constitutes a topological subgroup of the topological group of self-homeomorphisms of X equipped with the compact-open topology. Clearly, there is a continuous homomorphism $h\colon G \to H$ given by $h(g)(x) := \alpha(g,x)$ for $g \in G$ and $x \in X$. Furthermore, $\beta\colon H \times X \to X$ defined by $\beta(g,x) := g(x)$ for $g \in H$ and $x \in X$ is a continuous action of H on X. Finally, we are left to note that evidently $\alpha(g,x) = \beta(h(g),x)$ for all $g \in G$ and $x \in X$. $\square$

Next we want to relate Definition 4.1 to the concepts of strong periodicity and strong chaos introduced in [17,18].

Definition 4.3 Let G be a topological group and α a continuous action of G on some topological space X. Then the action α is said to be *strongly periodic* if the set $\{g \in G \mid \forall y \in X : \alpha(g,y) = y\}$ is syndetic in G. A point $x \in X$ is said to be *strongly periodic with respect to* α if the restricted action $\alpha_x\colon G \times \overline{\alpha(G,x)} \to \overline{\alpha(G,x)}$ is strongly periodic, i.e., $\{g \in G \mid \forall y \in \overline{\alpha(G,x)} : \alpha(g,y) = y\}$ is syndetic in G. Finally, α is called *strongly chaotic* if

(1) α is topologically transitive,

(2) the set of points being strongly periodic with respect to α is dense in X,

(3) α is not minimal.

Proposition 4.4 *Let G be a topological group and α a continuous action of G on some topological space X. Then α is $\mathcal{Q}(G)$-periodic if and only if α is strongly periodic.*

Generalizing a classical result by Banks et al. [2], in [17] it has been shown that the notion of strong chaos established in Definition 4.3 coincides with the one introduced by Devaney in [7] for the classical setting. That is to say, provided that G is one of the additive topological groups $\mathbb{R}$ or $\mathbb{Z}$, respectively, and α is a continuous action of G on some non-void metric space, then α is strongly chaotic if and only if α is chaotic in the sense of [7,13]. For a more detailed discussion on this, comprising the general level of continuous semigroup actions, we refer to [17].

Let us explore strong periodicity and strong chaos for the case where the acting topological group is discrete. As the second item of the subsequent remark reveals, there is just a slight difference between the concept of chaos introduced in [4] and the discrete instance of strong chaos.

Remark 4.5 [[18]] Let G be a discrete topological group and let α be a continuous action of G on some topological space X.

(a) A point $x \in X$ is strongly periodic with respect to α if and only if $\alpha(G,x)$ is finite.

(b) The action α is strongly chaotic if and only if α is not minimal and α is chaotic in the sense of [4], i.e., α is topologically transitive and

$$\{x \in X \mid \alpha(G,x) \text{ is finite}\}$$

is dense in X.

As mentioned in the introduction, we want to characterize those classes $\mathcal{D}$ of compactifications of a (locally compact Hausdorff) topological group G for which G admits a faithful $\mathcal{D}$-chaotic continuous action on some (compact) Hausdorff space. In fact, we are already prepared to conduct the first essential step towards the desired characterization result: we show that, if G admits a faithful $\mathcal{D}$-chaotic continuous action on some Hausdorff space, then $\mathcal{D}$ separates compact subsets in G. The idea of the subsequent proof originates from [18, Proposition 1].

Proposition 4.6 *Let G be a topological group, $\mathcal{D} \subseteq \mathcal{C}(G)$ a non-empty subclass, and α a faithful topologically transitive continuous action of G on some Hausdorff space X. If the set of points being $\mathcal{D}$-periodic with respect to α is dense in X, then $\mathcal{D}$ separates compact subsets in G.*

Proof. Let $K \subseteq G \setminus \{e_G\}$ be compact. Without loss of generality, assume K to be non-empty. Since X is Hausdorff and α is faithful and continuous, for each $g \in K$, there exist an open neighborhood U of g in G as well as two disjoint non-empty open subsets $V, W \subseteq X$ such that $\alpha(U, V) \subseteq W$. By compactness of K, there is a non-void family of non-empty open subsets $U_0, \ldots, U_{n-1} \subseteq G$, $V_0, \ldots, V_{n-1} \subseteq X$, $W_0, \ldots, W_{n-1} \subseteq X$ where $K \subseteq \bigcup_{i=0}^{n-1} U_i$, $V_i \cap W_i = \emptyset$ and $\alpha(U_i, V_i) \subseteq W_i$ for all $i \in \{0, \ldots, n-1\}$. Due to Lemma 2.1, we may find an open non-empty subset $S \subseteq X$ and elements $g_0, \ldots, g_{n-1} \in G$ such that $\alpha(g_i, S) \subseteq V_i$ for all $i \in \{0, \ldots, n-1\}$. By assumption, there exists some point $x \in S$ being $\mathcal{D}$-periodic with respect to α. This means that there exist $(H, h) \in \mathcal{D}$ and a continuous action β of H on $\overline{\alpha(G, x)}$ such that $\alpha(g, y) = \beta(h(g), y)$ for all $g \in G$ and $y \in \overline{\alpha(G, x)}$. We are left to show that $\ker(h) \cap K = \emptyset$. To this end, let $g \in K$. Then there exists some $i \in \{0, \ldots, n-1\}$ such that $g \in U_i$. As $\alpha(g_i, x) \in V_i$, $\alpha(U_i, V_i) \subseteq W_i$ and $V_i \cap W_i = \emptyset$, it follows that $\alpha(g, \alpha(g_i, x)) \neq \alpha(g_i, x)$, i.e., $\beta(h(g_i^{-1} g g_i), x) = \alpha(g_i^{-1} g g_i, x) \neq x$. Consequently, $g_i^{-1} g g_i \notin \ker(h)$ and therefore $g \notin \ker(h)$. This proves our claim and thus shows that $\mathcal{D}$ separates compact subsets in G. □

5 Chaotic actions on functional spaces

As established by Lemma 2.2 and Proposition 4.6, if G is a topological group and $\mathcal{D}$ is a non-empty subclass of $\mathcal{C}(G)$ such that G admits a faithful $\mathcal{D}$-chaotic continuous action on some Hausdorff space, then G is non-compact and $\mathcal{D}$ separates compact subsets in G. In this section we prove that the converse is valid for locally compact Hausdorff topological groups.

To this end, let us first briefly recall the well-known concept of a Haar functional. Let G be a locally compact Hausdorff topological group. Then there exists a *right Haar functional on* G, i.e., a non-trivial positive linear functional $I \colon C_c(G) \to \mathbb{R}$ where $I(f \circ \rho_G(g)) = I(f)$ for all $f \in C_c(G)$ and $g \in G$. Furthermore, if I and J are right Haar functionals on G, then there exists $r \in (0, \infty)$ such that $J(f) = rI(f)$ for all $f \in C_c(G)$. Assume I to be a right Haar functional on G. We obtain a normed $\mathbb{R}$-vector space by equipping $C_c(G)$ with the obvious point-wise operations and the norm given by

$$\|f\|_I := I(|f|) \quad (f \in C_c(G)).$$

Moreover, $T_I\colon (C_b(G), \|\cdot\|_\infty) \to (C_c(G), \|\cdot\|_I)'$ given by

$$T_I(f)(q) := I(fq) \quad (f \in C_b(G),\, q \in C_c(G))$$

is a one-to-one continuous linear operator and $\|T_I\| \leq 1$. For more details on this particular topic, we refer to standard textbooks, such as [11,3,6]. In the next step we want to introduce two canonical continuous actions arising from a given Haar functional in a quite natural manner.

Let G be a locally compact Hausdorff topological group and let I be a right Haar functional on G. We consider the closed dual unit ball $B_I(G) := B((C_c(G), \|\cdot\|_I)')$ endowed with the weak-* topology, i.e., the initial topology with respect to the maps $B_I(G) \to \mathbb{R}$, $l \mapsto l(f)$ where $f \in C_c(G)$. Since we do not expect any confusion to be caused by this, we shall not distinguish between the set $B_I(G)$ and the topological space $B_I(G)$. Recall that $B_I(G)$ is a compact Hausdorff space due to the celebrated Banach-Alaoglu Theorem. Let us define $\varphi_I\colon G \times B_I(G) \to B_I(G)$ by

$$\varphi_I(g,l)(f) := l(f \circ \rho_G(g^{-1})) \quad (g \in G,\, l \in B_I(G),\, f \in C_c(G)).$$

Furthermore, we denote by $S_I(G)$ the closure of $T_I[B(C_b(G), \|\cdot\|_\infty)]$ in $B_I(G)$ (with respect to the weak-* topology). As concerning $B_I(G)$, we shall not distinguish between $S_I(G)$ as a set and $S_I(G)$ as a topological subspace of $B_I(G)$.

In order to substantiate that the action introduced above is indeed continuous (see Proposition 5.2), we need to recall the following basic lemma, which asserts that every compactly supported continuous real-valued function on a topological group is uniformly continuous with regard to the respective left uniform structure.

Lemma 5.1 (e.g. [6]) *Let G be a topological group and let $f \in C_c(G)$. For every $\varepsilon > 0$, there exists an open neighborhood of e_G in G such that $\|f - (f \circ \rho_G(g))\|_\infty \leq \varepsilon$ for all $g \in U$.*

Proposition 5.2 *Let G be a locally compact Hausdorff topological group and let I be a right Haar functional on G. Then φ_I constitutes a faithful non-minimal continuous action of G on $B_I(G)$. Furthermore,*

$$\varphi_I(g, T_I(f)) = T_I(f \circ \rho_G(g))$$

for all $g \in G$ and $f \in C_b(G)$ with $\|f\|_\infty \leq 1$. Hence, $\varphi_I(G, S_I(G)) = S_I(G)$. The restriction $\psi_I\colon G \times S_I(G) \to S_I(G)$ is a faithful non-minimal continuous action of G on $S_I(G)$.

Proof. First we substantiate that φ_I is well defined. To this end, let $g \in G$ and $f \in C_c(G)$. Evidently, $f \circ \rho_G(g^{-1})$ is continuous and $\operatorname{spt}(f \circ \rho_G(g^{-1})) = (\operatorname{spt} f)g$ is compact. Hence, $f \circ \rho_G(g^{-1})$ is an element of $C_c(G)$. This means that the expression defining φ_I is reasonable. Let $l \in B_I(G)$. Of course, $\varphi_I(g,l)$ is linear, and

$$|\varphi_I(g,l)(f)| = |l(f \circ \rho_G(g^{-1}))| \leq \|l\| \|f \circ \rho_G(g^{-1})\|_\infty \leq \|f \circ \rho_G(g^{-1})\|_\infty = \|f\|_\infty$$

for $f \in C_c(G)$. Thus, $\varphi_I(g,l) \in B_I(G)$. To show continuity, let $g_0 \in G$, $l_0 \in B_I(G)$,

$f \in C_c(G)$ and $\varepsilon > 0$. Then $V := \{l \in B_I(G) \mid |(l_0 - l)(f \circ \rho_G(g_0^{-1}))| < \varepsilon/2\}$ is an open neighborhood of l in $B_I(G)$. Besides, by Lemma 5.1, there exists an open neighborhood U of e_G in G such that $\|f - (f \circ \rho_G(g))\|_\infty \leq \varepsilon/2$ for every $g \in U$. Therefore, for all $g \in U^{-1}g_0$ and $l \in V$, we conclude that

$$\begin{aligned}|(\varphi_I(g_0, l_0) - \varphi_I(g, l))(f)| &\leq |(\varphi_I(g_0, l_0) - \varphi_I(g_0, l))(f)| + |(\varphi_I(g_0, l) - \varphi_I(g, l))(f)| \\ &= |(l_0 - l)(f \circ \rho_G(g_0^{-1}))| + |l((f \circ \rho_G(g_0^{-1})) - (f \circ \rho_G(g^{-1})))| \\ &< \varepsilon/2 + \|(f \circ \rho_G(g_0^{-1})) - (f \circ \rho_G(g^{-1}))\|_\infty \\ &= \varepsilon/2 + \|f - (f \circ \rho_G(g_0 g^{-1}))\|_\infty \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

This shows that φ_I is continuous. In fact, φ_I constitutes a continuous action of G on $B_I(G)$ because $\varphi_I(e_G, l)(f) = l(f \circ \rho_G(e_G)) = l(f)$ and

$$\begin{aligned}\varphi_I(g_0 g_1, l)(f) &= l(f \circ \rho_G(g_1^{-1} g_0^{-1})) = l(f \circ \rho_G(g_0^{-1}) \circ \rho_G(g_1^{-1})) \\ &= \varphi_I(g_1, l)(f \circ \rho_G(g_0^{-1})) = \varphi_I(g_0, \varphi_I(g_1, l))(f)\end{aligned}$$

for all $g_0, g_1 \in G$, $l \in B_I(G)$ and $f \in C_c(G)$. Evidently, φ_I is non-minimal because

$$\overline{\varphi_I(G, l_0)}^{B_I(G)} = \{l_0\} \neq B_I(G)$$

concerning the zero functional $l_0 \colon C_c(G) \to \mathbb{R}$, $f \mapsto 0$. Moreover, for all $g \in G$, $q \in C_c(G)$ and $f \in C_b(G)$ with $\|f\|_\infty \leq 1$, we obtain

$$\begin{aligned}\varphi_I(g, T_I(f))(q) &= T_I(f)(q \circ \rho_G(g^{-1})) = I(f \cdot (q \circ \rho_G(g^{-1}))) \\ &= I((f \circ \rho_G(g)) \cdot q) = T_I(f \circ \rho_G(g))(q).\end{aligned}$$

Since φ_I is a continuous action of G of $B_I(G)$, this equality readily implies that $\varphi_I(G, S_I(G)) = S_I(G)$, wherefore ψ_I is a well-defined continuous action of G on $S_I(G)$. As above, ψ_I is non-minimal because

$$\overline{\psi_I(G, l_0)}^{S_I(G)} = \{l_0\} \neq S_I(G)$$

concerning the zero functional $l_0 \colon C_c(G) \to \mathbb{R}$, $f \mapsto 0$. We are left to prove that ψ_I is faithful, which will readily imply faithfulness of φ_I as well. For this purpose, let $g \in G \setminus \{e_G\}$. Since G is a locally compact Hausdorff topological group, there exists an open relatively compact neighborhood U of e_G in G such that $g \notin \overline{U}\,\overline{U}^{-1}$, i.e., $\overline{U} \cap g^{-1}\overline{U} = \emptyset$. Since G is a locally compact Hausdorff topological group and thus normal (see [1]), the Urysohn Lemma asserts the existence of continuous functions $f, q \colon G \to \mathbb{R}$ such that $f[G] \subseteq [0, 1]$, $f[\overline{U}] = \{1\}$, $f[\overline{U}g] = \{0\}$, and $q[G] \subseteq [0, \infty)$, $e_G \in q^{-1}[(0, \infty)] \subseteq U$. Obviously, f is bounded and $\|f\|_\infty = 1$. As $\operatorname{spt} q \subseteq \overline{U}$, it follows that q has compact support and that $fq = q$. Moreover, $\operatorname{spt}(q \circ \rho_G(g^{-1})) = (\operatorname{spt} q)g \subseteq \overline{U}g$ and hence $f \cdot (q \circ \rho_G(g^{-1})) = 0$. We conclude that $\psi_I(g, T_I(f))(q) = I(f \cdot (q \circ \rho_G(g^{-1}))) = 0$, whereas $T_I(f)(q) = I(fq) = I(q) > 0$. Thus, $\psi_I(g, T_I(f)) \neq T_I(f)$. This shows that ψ_I is faithful. □

For the rest of this section, we are concerned with proving that, if G is a non-compact locally compact Hausdorff topological group and $\mathcal{D} \subseteq \mathcal{C}(G)$ separates compact subsets in G, then ψ_I is $\mathcal{D}$-chaotic. Towards this aim there are essentially two steps to be taken: the first one will be to establish topological transitivity (Proposition 5.3), the second consists of showing density of $\mathcal{D}$-periodic points (Proposition 5.4).

As the proof of Proposition 5.3 reveals, topological transitivity of ψ_I is basically a consequence of the Tietze Theorem, whereas topological transitivity of φ_I follows from the Hahn-Banach Theorem. Although the latter observation shall not be needed for any of the subsequent considerations, we want to present its proof here – just to point out the structural similarity to the proof of the former fact.

Proposition 5.3 *Let G be a locally compact Hausdorff topological group and let I be a right Haar functional on G. The following are equivalent:*

(a) φ_I is topologically transitive.

(b) ψ_I is topologically transitive.

(c) G is not compact.

Proof. (a)$\vee$(b)$\Longrightarrow$(c) Assume G to be compact, i.e., $1_G \in C_c(G)$. Let $c := I(1_G)$. Clearly, $U_0 := \{l \in B_I(G) \mid |l(1_G)| < c/2\}$ and $U_1 := \{l \in B_I(G) \mid |l(1_G) - c| < c/2\}$ are disjoint open subsets of $B_I(G)$. Since $T_I(0_G) \in U_0$ and $T_I(1_G) \in U_1$, it follows that $S_I(G) \cap U_i \neq \emptyset$ for each $i \in \{0,1\}$. Moreover, it is straightforward to check that $\varphi_I(G, U_i) = U_i$ for each $i \in \{0,1\}$. Therefore, $\varphi_I(G, U_0) \cap U_1 = \emptyset$. Thus, neither φ_I nor ψ_I is topologically transitive.

(c)$\Longrightarrow$(a) Suppose that G is not compact. Let U_0 and U_1 be non-empty open subsets of $B_I(G)$. There exist $l_0, l_1 \in B_I(G)$, $Q \subseteq C_c(G)$ finite and $\varepsilon > 0$ such that $V_i := \bigcap_{q \in Q} \{l \in B_I(G) \mid |(l_i - l)(q)| < \varepsilon\} \subseteq U_i$ for each $i \in \{0,1\}$. We show that $\varphi_I(G, V_0) \cap V_1 \neq \emptyset$, which implies that $\varphi_I(G, U_0) \cap U_1 \neq \emptyset$. To this end, we observe that $K := \bigcup \{\operatorname{spt} q \mid q \in Q\}$ and therefore $K^{-1}K$ are compact in G. Consequently, $K^{-1}K \neq G$ because G is non-compact. Let $g \in G \setminus K^{-1}K$ and consider the linear subspaces $A := \operatorname{lin} Q$ and $B := \operatorname{lin}\{q \circ \rho_G(g^{-1}) \mid q \in Q\}$ of $C_c(G)$. Since $Kg \cap K = \emptyset$ and therefore $A \cap B = \{0_G\}$, the Hahn-Banach Theorem asserts the existence of some $l \in B_I(G)$ where $l|_A = l_0|_A$ and $l|_B = \varphi_I(g^{-1}, l_1)|_B$, i.e., $\varphi_I(g, l)|_A = l_1|_A$. In particular, $l \in V_0$ and $\varphi_I(g, l) \in V_1$. Hence, φ_I is topologically transitive.

(c)$\Longrightarrow$(b) Suppose that G is not compact. Let U_0 and U_1 be non-empty open subsets of $S_I(G)$. There exist $f_0, f_1 \in C_b(G)$ with $\|f_0\|_\infty, \|f_1\|_\infty \leq 1$, $Q \subseteq C_c(G)$ finite and $\varepsilon > 0$ such that $V_i := \bigcap_{q \in Q} \{l \in S_I(G) \mid |(T_I(f_i) - l)(q)| < \varepsilon\} \subseteq U_i$ for each $i \in \{0,1\}$. We show that $\psi_I(G, V_0) \cap V_1 \neq \emptyset$, which implies that $\psi_I(G, U_0) \cap U_1 \neq \emptyset$. For this purpose, we first observe that $K := \bigcup \{\operatorname{spt} q \mid q \in Q\}$ and therefore $K^{-1}K$ are compact in G. Consequently, $K^{-1}K \neq G$ because G is non-compact. Let $g \in G \setminus K^{-1}K$. Since G is a locally compact Hausdorff topological group and therefore normal (see [1]) and $Kg \cap K = \emptyset$, we may apply the Tietze Theorem to conclude that there exists some $f \in C_b(G)$ with $\|f\|_\infty \leq 1$ such that $f|_K = f_0|_K$

and $f|_{Kg} = (f_1 \circ \rho_G(g^{-1}))|_{Kg}$. Evidently, it follows that $fq = f_0 q$ and

$$f \cdot (q \circ \rho_G(g^{-1})) = (f_1 \circ \rho_G(g^{-1})) \cdot (q \circ \rho_G(g^{-1})),$$

and so we conclude that $T_I(f)(q) = I(fq) = I(f_0 q) = T_I(f_0)(q)$ and

$$\begin{aligned}\psi_I(g, T_I(f))(q) &= T_I(f)(q \circ \rho_G(g^{-1})) = I(f \cdot (q \circ \rho_G(g^{-1}))) \\ &= I((f_1 \circ \rho_G(g^{-1})) \cdot (q \circ \rho_G(g^{-1}))) = I(f_1 q) = T_I(f_1)(q)\end{aligned}$$

for all $q \in Q$. In particular, $T_I(f) \in V_0$ and $\psi_I(g, T_I(f)) \in V_1$. Hence, ψ_I is topologically transitive. □

Our next objective is to take care of the density condition. We will see that a fair share of work has already been conducted in the proof of Proposition 3.5.

Proposition 5.4 *Let G be a locally compact Hausdorff topological group. If a subclass $\mathcal{D} \subseteq \mathcal{C}(G)$ separates compact subsets in G, then the set of points being $\mathcal{D}$-periodic with respect to ψ_I is dense in $S_I(G)$.*

Proof. Let $f \in C_b(G)$ where $\|f\|_\infty \leq 1$, let $Q \subseteq C_c(G)$ be finite, and let $\varepsilon > 0$. Define $\delta := \varepsilon/(1 + \sup_{q \in Q} \|q\|_I)$. Let us furthermore consider the compact subset $K := \bigcup\{\operatorname{spt} q \mid q \in Q\}$. By Proposition 3.5, there exist $(H, h) \in \mathcal{D}$ and $p \in C(H)$ with $\|p\|_\infty \leq 1$ such that $\sup_{x \in K} |f(x) - (p \circ h)(x)| \leq \delta$. If $q \in Q$, then

$$|T_I(f)(q) - T_I(p \circ h)(q)| = |I((f - (p \circ h))q)| \leq I(|f - (p \circ h)||q|) \leq \delta \|q\|_I \leq \varepsilon.$$

The only item remaining to be verified is that $T_I(p \circ h)$ is $\mathcal{D}$-periodic with respect to ψ_I. For this purpose, let us first argue that

$$\kappa\colon\ H \to (C_b(G), \|\cdot\|_\infty),\ g \mapsto p \circ \rho_H(g) \circ h$$

is continuous. To this end, let $g_0 \in H$ and $\varepsilon > 0$. Since H is a compact topological group and p is continuous, there exists an open neighborhood U of g_0 in H such that $|p(\rho_H(g_0)(x)) - p(\rho_H(g)(x))| = |p(xg_0) - p(xg)| < \varepsilon$ for all $g \in U$ and $x \in H$. That is, $\|(p \circ \rho_H(g) \circ h) - (p \circ \rho_H(g) \circ h)\|_\infty \leq \varepsilon$. Thus, κ is continuous. Let $E := T_I[\kappa[H]]$. We observe that $\psi_I(G, T_I(p \circ h)) = T_I[\kappa[h[G]]] \subseteq T_I[\kappa[H]] = E$. Since $T_I \circ \kappa$ is continuous and H is compact, E is compact and hence closed in $S_I(G)$. Thus, the closure of $\psi_I(G, T_I(p \circ h))$ in $S_I(G)$ must be contained in E as well. Conversely, $h[G]$ is dense in H, wherefore

$$E = T_I[\kappa[H]] \subseteq \overline{T_I[\kappa[h[G]]]}^{S_I(G)} = \overline{\psi_I(G, T_I(p \circ h))}^{S_I(G)}.$$

This substantiates that $\overline{\psi_I(G, T_I(p \circ h))}^{S_I(G)} = E$. Let us define $\beta\colon H \times E \to E$ by

$$\beta(g_0, T_I(p \circ \rho_H(g_1) \circ h)) := T_I(p \circ \rho_H(g_1) \circ \rho_H(g_0) \circ h) = T_I(p \circ \rho_H(g_0 g_1) \circ h)$$

for all $g_0, g_1 \in H$. Since T_I is one-to-one and $h[G]$ is dense in H, we conclude that β is well-defined by the expression above. Furthermore, β is continuous. To explain this, consider the continuous maps $\alpha\colon H \times H \to H$ and $\gamma\colon H \times H \to H \times E$ given by $\alpha(g_0, g_1) := g_0 g_1$ and $\gamma(g_0, g_1) := (g_0, T_I(\kappa(g_1)))$ for all $g_0, g_1 \in H$. Clearly,

$$\begin{aligned} T_I(\kappa(\alpha(g_0, g_1))) &= T_I(\kappa(g_0 g_1)) = T_I(p \circ \rho_H(g_0 g_1) \circ h) \\ &= \beta(g_0, T_I(\kappa(g_1))) = \beta(\gamma(g_0, g_1)) \end{aligned}$$

for all $g_0, g_1 \in G$. Since γ is onto, $H \times H$ is compact, and $H \times E$ is Hausdorff, it follows that γ is a quotient map. As we have seen above, $\beta \circ \gamma = T_I \circ \kappa \circ \alpha$ and therefore β is continuous. In fact, β constitutes a continuous action of H on E. Finally, we are left to note that

$$\begin{aligned} \psi_I(g_0, T_I(\kappa(g_1))) &= \psi_I(g_0, T_I(p \circ \rho_H(g_1) \circ h)) = T_I(p \circ \rho_H(g_1) \circ h \circ \rho_G(g_0)) \\ &= T_I(p \circ \rho_H(g_1) \circ \rho_H(h(g_0)) \circ h) = \beta(h(g_0), T_I(\kappa(g_1))) \end{aligned}$$

for all $g_0 \in G$ and $g_1 \in H$. $\square$

Everything is prepared to state and prove the main result of the paper.

Theorem 5.5 *Let G be a locally compact Hausdorff topological group and $\mathcal{D} \subseteq \mathcal{C}(G)$ a non-empty subclass. The following are equivalent:*

(1) G is non-compact and $\mathcal{D}$ separates compact subsets in G.

(2) G admits a faithful $\mathcal{D}$-chaotic continuous action on some Hausdorff space.

(3) G admits a faithful $\mathcal{D}$-chaotic continuous action on some compact Hausdorff space.

(4) ψ_I is $\mathcal{D}$-chaotic.

Proof. (1) $\Longrightarrow$ (4) by Proposition 5.2, Proposition 5.3, and Proposition 5.4.

(4) $\Longrightarrow$ (3) due to Proposition 5.2 and the fact that $S_I(G)$ is closed in the compact Hausdorff space $B_I(G)$ and therefore itself constitutes a compact Hausdorff space.

(3) $\Longrightarrow$ (2) trivially.

(2) $\Longrightarrow$ (1) according to Proposition 4.6 and Lemma 2.2. $\square$

In particular, Theorem 5.5 yields the following: a locally compact Hausdorff topological groups admits a faithful strongly chaotic continuous action on some (compact) Hausdorff space if and only if it is strongly syndetically separated and not compact. This result is partially due to [18, Theorem 1]. However, in [18] just the equivalence of (1) and (2) is established, but there condition (3) is not considered and the continuous group action constructed to argue that (1) implies (2) does not live on a compact phase space. In this respect Theorem 5.5 constitutes a substantial improvement of [18, Theorem 1]. Furthermore, we observe that, combined with Remarks 3.11 and 4.5, the previous theorem specializes to a characterization of infinite residually finite groups, which appeared as a main result in [4, Theorem 1]. Another instance of Theorem 5.5 reads as follows:

Corollary 5.6 *Let G be a locally compact Hausdorff topological group. Then G is* *non-compact and maximally almost periodic if and only if G admits a faithful topologically transitive non-minimal continuous action α on some compact Hausdorff space X such that the set of points being Kronecker with respect to α is dense in X.*

Proof. This is an immediate consequence of Theorem 5.5, Proposition 3.6, and Proposition 4.2. □

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